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*Full Length Research Paper*

# Derivation of Klein-Gordon equation from Maxwell's electric wave equation

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**Maxwell's equation for electric field was used to derive Einstein energy-momentum relation. This was done by using Plank photon energy relation beside wave solution in insulating no charged matter. Klein-Gordon quantum equation was also derived from the same Maxwell's equation by utilizing resemblance between electric field vector and wave function in the intensity expression. However, the relation between polarization and electron rest mass was also used.**

**Key words:** Klein-Gordon equation, Maxwell's wave equation, photon mass.

## INTRODUCTION

It is known that Maxwell's equations with time derivative are invariant under relativistic Lorentz transforms. It is worth noting that relativistic Lorentz transforms are inapplicable to the time-harmonic waveguide waves in the frequency domain studied more than a century. The Klein-Gordon equation (Klein–Fock–Gordon equation or sometimes Klein–Gordon–Fock equation) is a relativistic version of the Schrödinger equation. It is the equation of motion of a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles. It cannot be straightforwardly interpreted as a Schrödinger equation for a quantum state, because it is second order in time and because it does not admit a positive definite conserved probability density. Still, with the appropriate interpretation, it does describe the quantum amplitude for finding a point particle in various places, the relativistic wavefunction, but the particle propagates both forwards and backwards in time.

Any solution to the Dirac equation is automatically a solution to the Klein–Gordon equation, but the converse is not true. The equation was named after the physicists, Oskar Klein and Walter Gordon, who in 1926 proposed that it describes relativistic electrons. Other authors making similar claims in that same year were Vladimir Fock, Johann Kudar, Théophile de Donder and Frans-H. van den Dungen, and Louis de Broglie (Sudoku, 2001; Kraus and Fleisch, 1999). Although it turned out that the Dirac equation describes the spinning electron, the Klein–Gordon equation correctly describes the spinless pion.

The problem of finding exact solutions of the Klein-Gordon equation for a number of special potentials has been a line of great interest in recent years. Some authors, by using different methods, studied the bound states of the Klein-Gordon equation under the condition that each of the scalar potentials is equal to its vector potential (Sudoku, 2001; Kraus and Fleisch, 1999; Fleisch, 2008; Halliday and Resnick, 1978; Clark, 1865; Lal, 1965).

At that time many scientists - one of them has been Maxwell himself - were convinced, that the correct notion for electrodynamics must be possible with quaternions. Conceptually, Maxwell's equations describe how electric charges and electric currents act as sources for the electric and magnetic fields. Further, it describes how a time varying electric field generates a time varying magnetic field and vice versa. Maxwell's equations have two major variants. The set of Maxwell's equations used

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total charge and total current including the difficult ones to calculate atomic level charges and currents in materials. Out of the four equations of Maxwell, the first two equations are scalar equations, while the last two are vector equations. Gauss' law for electricity is more commonly simply referred to as Gauss' law. Gauss' law for magnetism is remarkably similar to Gauss' law for electricity in form, but means something rather different (Vanderlinde, 2005; Nikon and Parker, 1978; Abbot, 1977).

The quantum mechanical equations rely heavily on Plank photon hypotheses. Despite the fact that the photon concept is related to quantum mechanics, Maxwell's equations which are related to the electromagnetic waves are isolated from photon concept, special relativity (SR) and quantum mechanics. Different attempts were made to bridge the gap between M.Es, SR and quantum mechanics (Parragh, 1975; Gatehouse, 1981; Mrato, 1991; Bruce and Minning, 1993; Rosen, 1994; Guo and Ma, 2001; Morgan, 2005; Kirchanov, 2012). Unfortunately these attempts need not unify the three equations in one step. Thus, there is a need to relate them to each other. This is done in the study's "Derivation of Einstein Equation from Maxwell's Equations and Derivation of Klein-Gordon Equation" where Einstein energy-momentum relation is derived from Maxwell's equations, and Klein–Gordon equation is also derived from the same Maxwell's equation. Subsequently, this study's focus is devoted for deriving Maxwell electric field equation. However, discussion and conclusion are shown as the study proceeds.

## MAXWELL'S ELECTRIC WAVE EQUATION

The M.Es used to describe the behavior of electromagnetic waves are (Wolf, 1976; Griffths, 1999):

$$\nabla . D=\rho,\ \nabla . B=0,\ \nabla\times E=-\frac{\partial B}{\partial t},\ \nabla\times H=J+\frac{\partial D}{\partial t} \qquad \text{……………………. (1)}$$

where D, B, E, H and J represent the electric flux density, the magnetic flux density, the electric field and the current density, respectively. Satisfying the following relations, we have (Salih and Teich, 2007):

$$B=\mu_0 H, J=\sigma E, D=\varepsilon_0 E+P \qquad \text{………………. (2)}$$

where P, $\mu_0$ and $\varepsilon_0$ are the macroscopic polarization of the medium, the permittivity of free space and the permeability of free space, respectively. Applying the curl operator to both sides of the 3rd equation in (1), the following equation is obtained:

$$\nabla\times(\nabla\times E)=-\nabla\times\frac{\partial B}{\partial t}=-\frac{\partial}{\partial t}(\nabla\times B) \qquad \text{……… (3)}$$

Using the identity (Salih and Teich, 2007; Taylor, 2005):

$$\nabla\times(\nabla\times E)=\nabla(\nabla . E)-\nabla^2 E \qquad \text{…………………… (4)}$$

Equation (3) gives:

$$\nabla(\nabla . E)-\nabla^2 E=-\frac{\partial}{\partial t}(\nabla\times B) \qquad \text{……………………. (5)}$$

From (2) since:

$$B=\mu_0 H \qquad \text{…………………… (6)}$$

Then (5) becomes:

$$\nabla(\nabla . E)-\nabla^2 E=-\frac{\partial}{\partial t}(\nabla\times\mu_0 H) \qquad \text{……………………. (7)}$$

From equation (7), since:

$$\nabla\times H=J+\frac{\partial D}{\partial t} \qquad \text{…………………… (8)}$$

From (1) we have:

$$\nabla(\nabla . E)-\nabla^2 E=-\frac{\partial}{\partial t}\left(\mu_0 J+\mu_0\frac{\partial D}{\partial t}\right) \qquad \text{……………. (9)}$$

But:

$$D=\varepsilon_0 E+P \qquad \text{………………….. (10)}$$

Therefore:

$$\nabla(\nabla . E)-\nabla^2 E=-\mu_0\frac{\partial J}{\partial t}-\varepsilon_0\mu_0\frac{\partial^2 E}{\partial t^2}-\mu_0\frac{\partial^2 P}{\partial t^2} \qquad \text{……….. (11)}$$

Also:

$$J=\sigma E \qquad \text{……………………... (12)}$$

Then:

$$\nabla(\nabla . E)-\nabla^2 E+\mu_0\frac{\partial J}{\partial t}+\varepsilon_0\mu_0\frac{\partial^2 E}{\partial t^2}=-\mu_0\frac{\partial^2 P}{\partial t^2} \qquad \text{…… (13)}$$

The polarization, $P$, thus acts as a source term in the equation for radiation field (Hand and Finch, 1998).
Since:

$$D=\varepsilon_0 E, \nabla . D=\rho, \rho=0 \qquad \text{…………………….. (14)}$$

Therefore:

$$\varepsilon\nabla.E = \rho = 0, \nabla.E = 0 \quad \text{……………………..….. (15)}$$

Therefore equation (13) becomes:

$$-\nabla^2 E + \mu_0 \frac{\partial J}{\partial t} + \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2} \quad \text{……………. (16)}$$

This represents the wave equation for electric field.

Klein-Gordon equation for free particles is usually derived by using Einstein relativistic energy equation:

$$E^2 = p^2 c^2 + m_o{}^2 c^4 \quad \text{…..…....…………...… (17)}$$

where $E, p$ and $m_o$ are the energy, momentum and rest mass, respectively.

This equation is then multiplied by the wave function $\psi$ to get:

$$E^2\psi = p^2 c^2 \psi + m_o{}^2 c^4 \psi \quad \text{…..…....…………...… (18)}$$

The energy and momentum terms are replaced by considering the particles as free waves having the wave function:

$$i\hbar \frac{\partial \psi}{\partial t} = E\psi \quad \psi = Ae^{\frac{i}{\hbar}(px - Et)} \quad \text{…..…....…………...… (19)}$$

This equation is differentiated with respect to $t$ and $x$ to get:

$$i\hbar \frac{\partial \psi}{\partial t} = E\psi \quad -\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = E^2 \psi \quad \text{....……………..… (20a)}$$

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial x} = \frac{\hbar}{i}\nabla\psi = p\psi \quad -\hbar^2\nabla^2\psi = p^2\psi \quad \text{……..… (20b)}$$

Inserting (20) in (18), the following equation is obtained:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} - c^2\hbar^2\nabla^2\psi + m_o{}^2 c^4 \psi \quad \text{…...………….....… (21)}$$

This is Klein-Gordon equation.

## DERIVATION OF EINSTEIN EQUATION FROM MAXWELL'S EQUATIONS

Maxwell's equation for an electric of field intensity $E$ in a dielectric insulating non-charged medium material of electric dipole moment $P$ is given by equation (16) to be:

$$-\nabla^2 E + \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2} \quad \text{……......…………... (22)}$$

Where for non-charged insulating material:

$$\rho = 0, \sigma = 0$$

Where for simplification it is better to consider current density $J$ as a constant, that is:

$$\frac{\partial J}{\partial t} = 0 \quad \text{……………............ (23)}$$

The electric dipole moment is given by:

$$P = nq_0 x = \frac{Nq_0 x}{Ax}$$
$$= \frac{Q}{A} = \frac{\phi}{A} \quad \text{….........………..... (24)}$$
$$= D = \varepsilon E$$

Where n is the number density of charge, N is the total number, A is the area and x is the distance.

$V$ = Volume = $Ax$

$Q$ = Total charge = $Nq_0$

$q_0$ = Charge of a single pole according to Gauss law.

The charge $Q$ and total flux $\phi$ are related by:

$$Q = \phi \quad \text{………………………. (25)}$$

To solve equation (22), one can assume the electric field intensity in free space $E$ to be:

$$E = E_0 e^{i(kx - wt)} \quad \text{………………………. (26)}$$

Thus:

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E \quad \nabla^2 E = -k^2 E \quad \text{…………………. (27)}$$

From equations (26) and (24):

$$\frac{\partial^2 P}{\partial t^2} = -\mu_0 \varepsilon \omega^2 E \quad \text{………………………. (28)}$$

The speeds in vacuum $c$ and in the medium $v$ are given:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon}} \qquad \text{.......................... (29)}$$

Thus (28) reads:

$$-\mu_0 \frac{\partial^2 P}{\partial t^2} = -\mu_0 \varepsilon \omega^2 E = -\frac{1}{v^2}\omega^2 E = -\left(\frac{2\pi f}{f\lambda_m}\right)^2 E = -k_m{}^2 E \qquad \text{.................. (30)}$$

Inserting (27) and (30) in (22) yields:

$$k^2 - \frac{\omega^2}{c^2} = -k_m{}^2 \qquad \text{.......................... (31)}$$

Multiplying both sides by $c^2$ and $\hbar^2$, one gets:

$$c^2\hbar^2k^2 - \hbar^2\omega^2 = -c^2\hbar^2k_m{}^2 \qquad \text{.......................... (32)}$$

Using De Broglie and Plank hypotheses:

$$p = \frac{h}{\lambda} = \hbar k \quad E = hf = \hbar\omega \qquad \text{.......................... (33)}$$

Equation (32) can thus be given by:

$$c^2p^2 + c^2p_m{}^2 = E^2 \qquad \text{.......................... (34)}$$

Since the electromagnetic waves can be assumed as a photon moving with the speed of light $c$, the photon momentum rest mass $m_0$ is given by:

$$\hbar k_m = p_m = m_0 c \qquad \text{.......................... (35)}$$

Here the rest mass is assigned to a medium since the medium lower photon speed and it can even stop it when it is absorbed. Thus inserting (35) in (34) yields:

$$c^2p^2 + m_0{}^2c^4 = E^2 \qquad \text{.......................... (36)}$$

This is the Einstein expression that relates momentum to energy.

The derivation of this relation can be done by using the classical equation of energy and Plank hypothesis only. The classical energy for an electromagnetic wave photon oscillating particle with maximum velocity is given by:

$$E = \frac{1}{2}mv_m{}^2 \qquad \text{.......................... (37)}$$

Since for waves or any harmonic system, the root mean square (r. m. s) velocity $v_{rms}$ is given by:

$$v_{rms} = \frac{1}{\sqrt{2}}v_m \qquad \text{.......................... (38)}$$

By assuming the photon speed $c$ equal to the r. m. s speed, that is:

$$c = \frac{1}{\sqrt{2}}v_m \qquad \text{.......................... (39)}$$

It follows that:

$$E = m\left(\frac{v_m}{\sqrt{2}}\right)^2 = mc^2 \qquad \text{.......................... (40)}$$

According to Plank theory:

$$E = hf = \frac{hc}{\lambda} = mc^2 \qquad \text{.......................... (41)}$$

Therefore, the momentum $p$ is given by:

$$p = mc = \frac{mc^2}{c} = \frac{hc}{\lambda c} = \frac{h}{\lambda} \qquad \text{.......................... (42)}$$

**DERIVATION OF KLEIN-GORDON EQUATION**

The Klein-Gordon equation can be obtained by replacing the electric dipole moment term in equation (17) by the term standing for photon rest mass in equation (30) to get:

$$-\nabla^2 E + \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} = -k_m{}^2 E \qquad \text{........................ (43)}$$

Multiplying both sides by $c^2\hbar^2$ and using equation (29), the following equation is obtained:

$$-c^2\hbar^2\nabla^2 E + \hbar^2 \frac{\partial^2 E}{\partial t^2} = -c^2\hbar^2k_m{}^2 E \qquad \text{............... (44)}$$

According to relation (35):

$$P_m{}^2 = \hbar^2 k_m{}^2 = m_0{}^2 c^2$$

Thus (44) reads:

$$-c^2\hbar^2\nabla^2 E + m_0{}^2 c^4 E = -\hbar^2 \frac{\partial^2 E}{\partial t^2} \quad \text{……………. (45)}$$

The incorporation of mass for photon in Maxwell's equations corresponds to adding the term $m_0 A^\mu A_\mu$ to the electromagnetic field lagrangian.

Since in the electromagnetic (e. m) theory the oscillating electric wave $E$ is related to its e. m, the energy or intensity is obtained according to the relation:

$$I \propto c\varepsilon_0 E^2 \quad \text{………………………. (46)}$$

And since the e. m intensity, when treated as a stream of photons of density $n$ is given by:

$$I \propto nhf \propto |\psi|^2\ hf \quad \text{………………………. (47)}$$

Where the photon density is related to the wave function $\psi$ according to the relation:

$$n = |\psi|^2 \quad \text{………………………. (48)}$$

Comparing (46) and (47) it follows that:

$$E^2 \Leftrightarrow |\psi|^2 \quad E \Leftrightarrow \psi \quad \text{………………………. (49)}$$

Thus the correspondence between $E$ and $\psi$ secure the replacement of $E$ by $\psi$ in equation (45) to get:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -c^2\hbar^2\nabla^2\psi + m_0{}^2 c^4 \psi \quad \text{………………. (50)}$$

This represents Klein-Gordon equation for free electron.

## DISCUSSION

In the study's derivation of Einstein equation from Maxwell's equations, Einstein energy momentum relation (35) is derived by using Maxwell's equation (17) for electric field besides using Plank hypothesis of photon energy. The contradiction between the famous Einstein energy relations (40) and the classical energy expression (37) is removed by relating the effective wave velocity to the maximum wave velocity.

In the study's derivation of Klein-Gordon equation, Klein-Gordon quantum equation (50) is obtained from Maxwell's equation (17) by replacing polarization dipole term by rest mass term. This is not surprising as far as electric dipole atoms oscillating with the same frequency was shown in the e. m field where $P \approx ex \approx \varepsilon E \approx e^{i\omega t}$.

At this stage, resonance occurred and the photon was absorbed and stopped to be at rest, with mass $m_0$. The resemblance between $E$ and $\psi$ as shown by equations (46, 47 and 49) secures replacement of $E$ by $\psi$ to get Klein-Gordon quantum equation for the free electron.

## CONCLUSION

The fact that Einstein momentum-energy relation and Klein-Gordon quantum equation is derived from Maxwell's equations shows a possibility of unifying the electromagnetic theory with both special relativity and quantum mechanics. Thus, one can bridge the gap between special relativity and quantum mechanics.

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