International Journal of Physical Sciences Vol. 1(4), pp. 056-061, August 2013
Available online at http://academeresearchjournals.org/journal/ijps
ISSN 2331-1827 ©2013 Academe Research Journals

Full Length Research Paper

Solving the fractional vibration equation using modified Riemann-Liouville fractional derivative

Qazi Mahmood Ul Hassan* and Syed Tauseef Mohyud-Din

Department of Mathematics, Faculty of Sciences, HITEC University, Taxila Cantt, Pakistan.

Aceepted 30 August, 2013

In this paper, a framework was presented to obtain the solutions to the fractional vibration equation by homotopy analysis method (HAM). The fractional derivative is described in modified Riemann-Liouville derivative. The results reveal that the proposed method is very effective and simple and leads to accurate, approximately convergent solutions to fractional vibration equation.

Key words: Homotopy analysis method, fractional calculus, fractional vibration equation, wave velocity, Mittag-Leffler function, modified Riemann-Liouville fractional derivative.

INTRODUCTION

In recent years, analysis of fractional differential equations, which are obtained from the classical differential equations in mathematical physics, engineering, vibration and oscillation by replacing the second order time derivative by a fractional derivative of order α satisfying 1 < α ≤ 2, have been a field of growing interest as evident from literature survey (Momani, 2005a, b; Momani and Ibrahim, 2007; Das, 2008; Momani et al., 2007; Odibat et al., 2008). Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

Recently, a new modified Riemann-Liouville left derivative is proposed by Jumarie (1993, 2006). Comparing with the classical caputo derivative, the definition of the fractional derivative is not required to satisfy higher integer-order derivative than α. Secondly, the α th derivative of a constant is zero. For these merits, Jumarie modified derivative was successfully applied in the probability calculus (Jumarie, 2009a) and fractional Laplace problem (Jumarie, 2009b).

In this paper, the homotopy analysis method (Liao, 2003a, b) was applied to solve the fractional vibration equation. With the present method, numerical results can be obtained by using a few iterations. The HAM contains the auxiliary parameter $\hbar$. which provides us with a simple way to adjust and control the convergence region of solution series for large value of t (Liao, 2009). Unlike other numerical methods, it gives low degree of accuracy for large values of t. Therefore, the HAM handles linear and inhomogeneous problems without any assumption and restriction (Hashim et al., 2009; Das and Gupta, 2011).

In this paper, the fractional vibration equation was considered by using homotopy analysis method. This fractional vibration equation is obtained by replacing the second time derivative term in the corresponding vibration equation by a fractional derivative of order α with 1 < α ≤ 2. The derivatives are understood in the modified Riemann-Liouville sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of α = 2, the fractional vibration equation reduces to the standard vibration equation.

Modified Riemann-Liouville derivative

It is assumed that $f : R \to R, x \to f(x)$ denotes a continuous (but not necessarily differentiable) function and the partition of $h > 0$ is in the interval (0, 1). Through the fractional Riemann Liouville integral, the following equations are shown:

*Corresponding author. E-mail: qazimahmood@yahoo.com.

$$ {}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi\,, \quad \alpha > 0 \tag{1} $$

The modified Riemann-Liouville derivative is defined as:

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-\xi)^{n-\alpha}(f(\xi)-f(0))\,d\xi, \tag{2} $$

where $x \in [0,1],\ n-1 \le \alpha < n$ and $n \ge 1$.

Jumarie's derivative is defined through the fractional difference:

$$ \Delta^{\alpha} = (FW-1)^{\alpha} f(x) = \sum_0^{\infty}(-1)^k \binom{\alpha}{k} f[x+(\alpha-k)h], \tag{3} $$

where $FWf(x) = f(x+h)$. Then the fractional derivative (Jumarie, 2009a) is defined as the following limit:

$$ f^{(\alpha)} = \lim_{h\to 0}\frac{\Delta^{\alpha} f(x)}{h^{\alpha}} \tag{4} $$

The proposed modified Riemann-Liouville derivative as shown in equation (2) is strictly equivalent to equation (4). Meanwhile, this study would introduce some properties of the fractional modified Riemann-Liouville derivative in equations (5) and (6):

(a) Fractional Leibniz product law:

$$ {}_0D_x^{\alpha}(uv) = u^{(\alpha)}v + uv^{(\alpha)} \tag{5} $$

(b) Fractional Leibniz formulation:

$$ {}_0I_x^{\alpha}\,{}_0D_x^{\alpha} f(x) = f(x) - f(0), \quad 0 < \alpha \le 1 \tag{6} $$

Therefore, the integration by part can be used during the fractional calculus:

$$ {}_aI_b^{\alpha} u^{(\alpha)} v = (uv)/_a^b - {}_aI_b^{\alpha} uv^{(\alpha)} \tag{7} $$

(c) Integration with respect to $(d\xi)^{\alpha}$.

Assuming $f(x)$ denotes a continuous $R \to R$ function, the following quality is used for the integral with respect to $(d\xi)^{\alpha}$:

$$ {}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-\xi)^{\alpha-1} f(\xi)\,d\xi, \quad 0 < \alpha \le 1 $$
$$ = \frac{1}{\Gamma(1+\alpha)}\int_0^x f(\xi)(d\xi)^{\alpha} \tag{8} $$

THE HOMOTOPY ANALYSIS METHOD (HAM)

This study considers the following differential equation:

$$ FD[u(x,t)] = 0, \tag{9} $$

where *FD* is a nonlinear operator for this problem, *x* and *t* denote an independent variables, and $u(x,t)$ is an unknown function.

In the frame of HAM, the following 0th-order deformation can be constructed:

$$ (1-q)L(U(x,t;q) - u_0(x,t)) = q\hbar H(x,t)FD(U(x,t;q)), \tag{10} $$

where $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, *L* is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$, and $U(x,t;q)$ is an unknown function of the independent variables *x, t* and *q*. Obviously, when $q = 0$ and $q = 1$, it holds that:

$$ U(x,t;0) = u_0(x,t), \text{ and } U(x,t;1) = u(x,t), \text{ respectively} \tag{11} $$

Using the parameter *q*, $U(x,t;q)$ is expanded in Taylor series as follows:

$$ U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\, q^m, \tag{12} $$

where: $u_m = \left.\frac{1}{m!}\frac{\partial^m U(t;q)}{\partial^m q}\right|_{q=0}$

Assuming that the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function $H(x,t)$ are selected such that the series (Equation 12) is convergent at $q = 1$, then due to Equation 12, the following equation is derived:

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t) \tag{13}$$

Let us define the vector:

$$\vec{u}_n(x,t)=\{u_0(x,t),u_1(x,t),\ldots,u_n(x,t)\}$$

Differentiating (10) m times with respect to the embedding parameter q, then setting $q=0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation:

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=\hbar H(x,t)R_m(\vec{u}_{m-1}), \tag{14}$$

where: $R_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}FD(U(t;q))}{\partial^{m-1}q}\bigg|_{q=0}$, and

$$\chi_m=\begin{cases}0 & m\le 1,\\ 1 & m>1.\end{cases}$$

Finally, for the purpose of computation, the HAM solution (13) will be approximated by the following truncated series:

$$\phi_m(t)=\sum_{k=0}^{m-1}u_k(t).$$

Fractional vibration equation

In this study, the fractional calculus version of the standard vibration equation was considered in one dimension as:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{c^2}\frac{\partial^\alpha u}{\partial t^\alpha},\quad r\ge 0,\ t\ge 0,\ 1<\alpha\le 2, \tag{15}$$

which constitute the relation between the radial velocity of $u(r,t)$ to the fractional time derivative of order α ($1<\alpha\le 2$) of $u(r,t)$ and c is the wave velocity of free vibration. It is easily seen that the whole hierarchy of moments $M_k=<r^k(t)>$ have the same time dependence as for the fractional Brownian motion though their statistical features are quite different. Now taking the Laplace transform of equation (15), we get:

$$s^\alpha\bar{u}(r,s)=c^2\left[\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}\right] \tag{16}$$

where $\bar{u}(r,s)=L[u(r,t)]$.

Equation (4) can be written as:

$$r\frac{d^2}{dr^2}\bar{u}(r,s)+\frac{d}{dr}\bar{u}(r,s)-\frac{s^\alpha}{c^2}r\bar{u}(r,s)=0, \tag{17}$$

Taking the series solution of $\bar{u}(r,s)$ as:

$$\bar{u}(r,s)=\sum_{n=0}^{\infty}a_n r^{n+\rho},\quad a_0\ne 0,\ \rho\ \text{is real} \tag{18}$$

This study finally obtain:

$$\bar{u}(r,s)=A(1+\ln r)+\frac{B}{c^2}s^\alpha r^2+o\left(s^{2\alpha}\right) \tag{19}$$

where A and B are constants.

Therefore, $u(r,t)\approx t^{-\alpha-1}$ (20)

It clearly exhibits the power law decay of $u(r,t)$ with α in contrast to the stretched exponential decay characteristic generally seen in fractional Brownian motion.

Solution of the problem by homotopy analysis method

Here, the application of the homotopy analysis method is discussed for solving the fractional vibration equation (15) with the initial conditions:

$$u(r,0)=r^2, \tag{21}$$

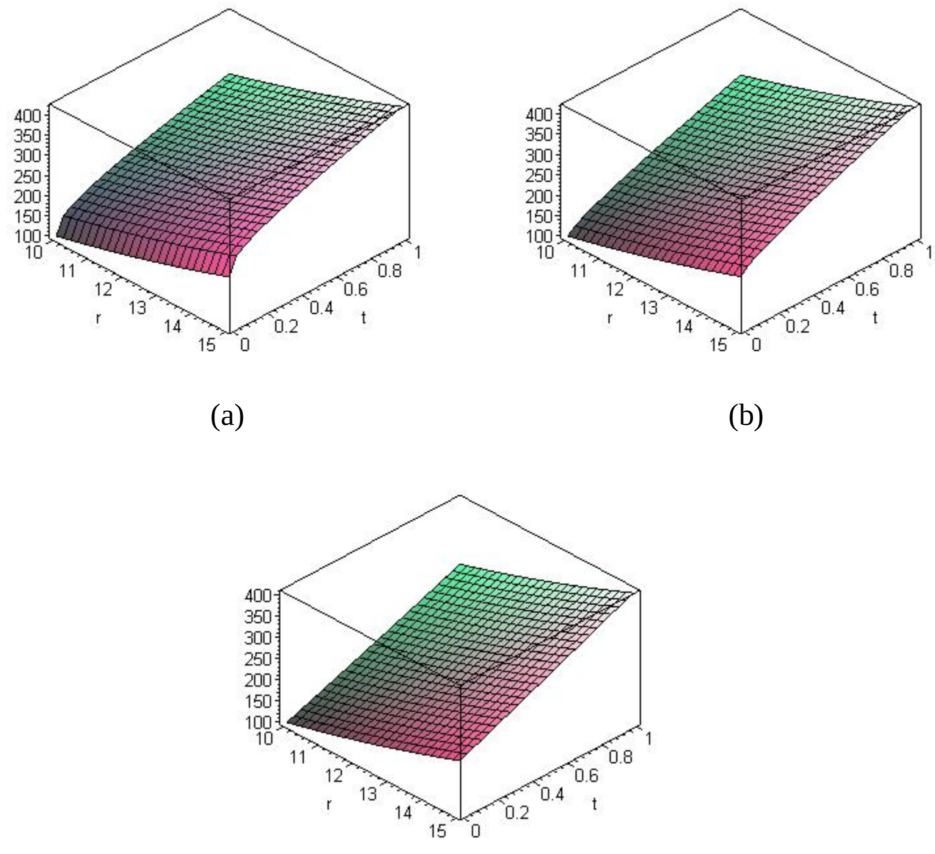
$$\frac{\partial}{\partial t}u(r,0)=cr, \tag{22}$$

Equation (15) can be written as:

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^{2-\alpha}}{\partial t^{2-\alpha}}\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right], \tag{23}$$

To solve Equations (21) to (23) by homotopy analysis method, according to (10), the 0th-order deformation can be given by:

$$(1-q)L(U(r,t;q)-u_0(r,t))=q\hbar H(r,t)\left(\frac{\partial^2 U(r,t;q)}{\partial t^2}-p\left(c^2\frac{\partial^{2-\alpha}}{r^{2-\alpha}}\left(\frac{\partial^2 U(r,t;q)}{\partial r^2}+\frac{1}{r}\frac{\partial U(r,t;q)}{\partial r}\right)\right)\right) \tag{24}$$



(a) (b) (c)

Figure 1. Plot of *u(r, t)* with respect to *r* and *t* at $c = 5$.
(a) $\alpha = 1/3$, (b) $\alpha = 2/3$, (c) $\alpha = 1$.

We can start with an initial approximation of $u_0(r,t) = 1$ and choose the auxiliary linear operator:

$$L(U(r,t;q)) = \frac{\partial^2 U(r,t;q)}{\partial t^2},$$

with the property: $L(c_1 + t.c_2) = 0$

where c_1, c_2 is an integral constant. We also choose the auxiliary function to be: $H(r,t) = 1.$

Hence, the *m*th-order deformation can be given by:

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar H(r,t) R_m(\vec{u}_{m-1}(r,t)), \tag{25}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2(u_{m-1}(r,t;q))}{\partial t^2} - \left(c^2 \frac{\partial^{2-\alpha}}{\partial r^{2-\alpha}}\left[\frac{\partial^2 U_{m-1}(r,t;q)}{\partial r^2} + \frac{1}{r}\frac{\partial U_{m-1}(r,t;q)}{\partial r}\right]\right) \tag{26}$$

Now the solution of the *m*th-order deformation equation (25) for $m \geq 1$ become:

$$u_m(r,t) = \chi_m u_{m-1}(r,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1}(r,t))]. \tag{27}$$

Consequently, the first few terms of the HAM series solution for $\hbar = -1$ are as follows:

$$u_0(r,t) = r^2 + c.r.t,,$$

$$u_1(r,t) = \frac{4.c^2 t^\alpha}{\Gamma(1+\alpha)},$$

$$u_2(r,t) = \frac{c^3 t^{\alpha+1}}{r\Gamma(\alpha+2)},$$

$$u_3(r,t) = \frac{c^5 t^{2\alpha+1}}{r^3\Gamma(2\alpha+2)},$$

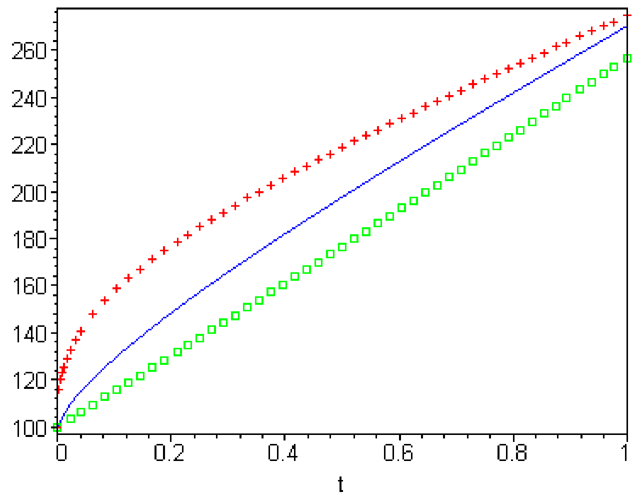


Figure 2. Plot of *u(r, t)* vs. *t* for different values of α at *r = 10* and *c = 5*. α : (+) α = 1/3 , (—) α =2/3, (□) α =1 .

.

.

.,

and so on. Hence, the HAM series solution is:

$$u(r,t)=u_0(r,t)+u_1(r,t)+u_2(r,t)+u_3(r,t)+..., \quad (28)$$

$$\mathrm{u}(r,t)$$

$$=r^2+crt+\frac{4c^2t^\alpha}{\Gamma(\alpha+1)}+\frac{c^3t^{\alpha+1}}{r\Gamma(\alpha+2)}+\frac{c^5t^{2\alpha+1}}{r^3\Gamma(2\alpha+2)}+\frac{9c^7t^{3\alpha+1}}{r^5\Gamma(3\alpha+2)}+...,$$

$$=r^2+\frac{4c^2t^\alpha}{\Gamma(\alpha+1)}+crtE_{\alpha,2}\left(\frac{c^2}{r^2}kt^\alpha\right),$$

where $k^n=\left[1.3.5.......(2n-3)\right]^2$ and $E_{\alpha,b}(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n\alpha+b)}$ is the generalized Mittag-Leffler function (Mittag-Leffler, 1904).

NUMERICAL RESULTS AND DISCUSSION

Here, the numerical results of the displacement for various values of radii of the membrane and time are presented in Figures 1 and 2. It is observed from Figure 1 that the displacement increases with the increase of both r and t for the wave velocities c=5. It is also seen from Figure 2 that the displacement rapidly increases with the increase of t and c both at a fixed value of the radius of the membrane (r = 10) but decreases with the increase of the fractional time derivative, which is in complete agreement with the fact described in “The homotopy analysis method (HAM)”. The numerical calculations and figures are made using Maple software (Version 17).

Conclusion

Homotopy analysis method is very powerful in finding the solutions for various physical, vibration and oscillation problems. The main interest is to construct a competitive study of finding numerical solutions to vibration equation. It is seen that the method used in this study is efficient for finding the solutions in higher degree of accuracy. This study’s method is direct and straightforward and it avoids the volume of calculations. Also, HAM facilitates computational work for which it gives the required solution faster when compared with the other methods.

REFERENCES

Das S (2008). Solution of Fractional Vibration Equation by the Variational Iteration Method and Modified Decomposition Method. Int. J. Nonlinear Sci. Numer. Simulat., 9: 361-365.

Das S, Gupta PK (2011). Application of homotopy perturbation method and homotopy analysis method to fractional vibration equation, 2011, 88(2), 430-441.

Hashim, Abdulaziz O, Momani S (2009). Homotopy analysis method for fractional IVPs. Commun. Nonlinear Sci. Numer. Simulat., 14: 674-684.

Jumarie G (1993). Stochastic differential equations with fractional Brownian motion input. Int. J. Syst. Sci., 6: 1113-1132.

Jumarie G (2006). New stochastic fractional models for Malthusian growth, the Poissonian birth process and optimal management of populations. Math. Comput. Model., 44: 231-254.

Jumarie G (2009a). Laplace s transform of fractional order via the Mittag –Leffler function and modified Riemann- Liouville derivative. Appl. Math. Lett., 22: 1659-1664.

Jumarie G (2009b). Table of some basic fractional calculus formulae derived from a modified Riemann - Liouville derivative for non-differentiable functions. Appl. Math. Lett., 22(2009).

Liao SJ (2003a). Beyond perturbation: Introduction to the homotopy analysis method, Chapman and Hall/CRC Press, Boca Raton, 2003.

Liao SJ (2003b). On the analytic solution of magneto hydro dynamic flows of non-Newtonian fluids over a stretching sheet. J. Fluid Mech., 488: 189-212.

Liao SJ (2009). Notes on the homotopy analysis method: some definitions and theorems. Commun. Nonlinear Sci. Numer. Simulat., 14: 983-997.

Mittag-Leffler GM (1904). Sopra la funzione Eα (x). R. Accad. Lincei, Rend. (Ser. V) 13: 3-5.

Momani S (2005a). An explicit and numerical solutions of the fractional KdV equation, Mathematics and Comput. Simul., Vol. 70(2): 110-118.

Momani S (2005b). Analytic and approximate solutions of the space- and time-fractional telegraph equations. Appl. Math. Comput., 170(2): 1126-1134.

Momani S, Ibrahim R (2007). Analytical solutions of a fractional oscillator by the decomposition method. Int. J. Pure Appl. Math., 37(1): 119-132.

Momani S, Odibat Z, Alawneh A (2007). Variational iteration method for solving the space- fractional and time- fractional KdV equation, Num. Meth. For Part. Diff. Equ. (2007) in press.

Odibat Z, Momani S, Alawneh A (2008). Analytic study on time-fractional Schrödinger equations: Exact solutions by GDTM, Journal of Physics: Conference Series 96 (2008) 012066.